FURTHER MATHEMATICS

Paper 9231/12
Paper 12

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should make use of results derived or given in earlier parts of a question.
- They should be able to recall and apply skills from the 9709 Syllabus where appropriate.

General comments

Most candidates presented their work clearly and logically, realising the need for unambiguous mathematical communication. Scripts from the strongest candidates showed knowledge across all topics. Gaps in knowledge were evident in some scripts.

Comments on specific questions

Question 1

(i) This was very well done and good knowledge of implicit differentiation was seen.

(ii) Many candidates were accurate in evaluating the second derivative, though a few made errors with signs.

Question 2

(i) Most candidates successfully applied the formula for $\cos P + \cos Q$ to the denominator. Better responses also applied the formula for $\cos P - \cos Q$ to fully justify the given identity.

(ii) This part was done to a high standard with candidates writing out enough terms to justify cancellation, and most remembering to give their answer in terms of N not n.

(iii) Better responses gave a valid reason for divergence by stating that $\cos N$ oscillates.

Question 3

This question was well answered with the majority of candidates successfully finding $\overrightarrow{PQ}$ in terms of two parameters and then forming simultaneous equations. Some assumed that $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ -1 \\ -6 \end{pmatrix}$ without justification.

Question 4

(i) Many candidates correctly used the idea of reverse differentiation. Reponses which involved the approach of integration by parts were less successful.

(ii) Almost all candidates applied integration by parts to I_n. Those who used the result from the previous part, or integrated x^n, successfully derived the given reduction formula. Others who tried to integrate e^{x^3} were less successful.

(iii) This part was well done with the majority of candidates accurately applying the reduction formula.

Question 5

(i) Most candidates accurately recalled the formula for surface area. Better responses fully simplified $\dot{x}$ and $\dot{y}$ before substituting them into the formula, which caused fewer errors and enabled a clear path to the given answer.

(ii) This part was well done with the majority of candidates accurately applying the given substitution and using the correct limits.

Question 6

(i) Almost all substituted for *x* into the original equation and cubed $y + 1$ to verify the result. The given equation in *y* was then used to correctly write down the value of S_3.

(ii) Most candidates used $\frac{\alpha^3\beta^3 + \beta^3\gamma^3 + \alpha^3\gamma^3}{\alpha^3\beta^3\gamma^3}$, with a few candidates opting to find a third cubic with reciprocal roots.

(iii) Several methods were employed in the final part, with a few candidates successfully working with the original equation in terms of *x*. Most candidates used the formula for the sum of squares and the given cubic in terms of *y* as intended. Some candidates who tried to recall complicated sigma formula made errors.

Question 7

Almost all candidates knew how to approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the constants and an occasional lack of clarity with notation. A few candidates used *x* as both a dependent and independent variable, and some gave expressions instead of equations as their answer.

Question 8

(i) Although most candidates clearly knew the requirement for an induction proof, sometimes details were omitted. Better responses checked the base case ($n = 1$ not $n = 2$), stated the assumption that the result is true for $n = k$ and proved the inductive step from the left hand side to the right hand side. They then summarised by stating that H_k implied H_{k+1}.

(ii) Almost all candidates knew that de Moivre's theorem related the series to the geometric progression in part **(i)**. However, it was common to see extra variables, such *m* or *n*, in the sum to infinity of the geometric progression which overcomplicated the expression and hindered progress. The strongest responses found the imaginary part of $\frac{-1}{2-1}$ or $\frac{z}{1-z}$ which led to the given answer.

Question 9

(i) Most candidates constructed the proof well although a few tried to square both sides or reversed the multiplication of **A** and **e**.

(ii) This part of the question was also well done, though a few candidates accepted zero eigenvectors without checking for errors in their working. Some candidates worked from **A**, others found **B** first, though the latter method produced more errors.

Question 10

(i) to (iv) Most candidates showed good knowledge of the algebraic and calculus skills involved, gaining all the marks available in the first four parts.

(v) Candidates found this part of the question more challenging. The better responses included sketches of the asymptotes meeting at a common point and positioned the branches correctly.

Question 11 – EITHER

(i) Some of the candidates who tackled this option found this part challenging, not appreciating the need to use $x = r\cos\theta$ for the distance from the line $\theta = \frac{1}{2}\pi$, and set $\frac{dx}{d\theta} = 0$ to find the maximum such distance. Most substituted appropriate values into $2\theta\tan\theta - 1$ and recorded a change in sign.

(ii) This part was very well done by the majority of candidates.

(iii) Most candidates sketched the correct shape of C_1 with the correct domain. Better responses had r strictly increasing and the correct intersection point with C_2.

(iv) Better responses correctly subtracted the area bounded C_2 and $\theta = \frac{1}{4}\pi$ from the area bounded C_1 and $\theta = \frac{1}{4}\pi$. Those who formed the wrong integral by having C_1 and C_2 interchanged in this subtraction often still gained credit for finding $\int \theta \sec^2\theta \, d\theta$ correctly using integration by parts.

Question 11 – OR

This was the more popular choice.

(i) (a) The majority of candidates were able to use row operations accurately, and better responses justified linear independence.

(b) Most candidates set $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ as a linear combination of the basis vectors and were able to show the given relationship directly. Some successfully used row operations. A few incorrectly assumed that $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ was in the null space.

(ii) It was pleasing to see how many candidates could solve this equation by separating the problem into two parts. Finding the particular solution and then identifying the basis for the null space was the most common method. A number of candidates combined the two parts and set up and solved equations or used the augmented matrix; working with equations did sometimes give rise to errors.

FURTHER MATHEMATICS

Paper 9231/22
Paper 22

Key messages

To score full marks in this paper candidates must be well versed in both Mechanics and Statistics. Any preference between these two areas can only be exercised in the choice of the final optional question.

When a result is given in a question, candidates must take care to give sufficient detail in their working, so that the Examiner is in no doubt that the offered solution is clear and complete. In all questions, however, candidates are advised to show all their working, as credit is given for method as well as accuracy.

In Mechanics questions, a diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although a candidate is always free to draw their own diagram as well.

General comments

Almost all candidates attempted all the compulsory questions, and some very good answers were frequently seen. Most candidates opted for the Statistics option in **Question 11**.

Previous reports have stressed the need for candidates to set out their work clearly, and this advice has been heeded by most. This was particularly important in the unstructured **Question 4**.

The rubric for this paper specifies that non-exact numerical answers be given to 3 significant figures. Candidates would therefore be well-advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one, as for example in **Question 9** and **Question 11**.

Comments on specific questions

Question 1

Candidates who knew the formulae for the radial and transverse components of acceleration usually scored full marks on this question. Many candidates either omitted the question or attempted to use formulae for constant acceleration.

Question 2

Since P is moving in simple harmonic motion, equating the ratio of the speeds at B and A to 2:1 using the standard result $v^2 = \omega^2\left(a^2 - x^2\right)$ leads to an equation for a^2 and hence the amplitude of the motion. Equating the maximum acceleration $\omega^2 a$ to the given value of 1 leads to the value of ω. Finally the speed at O is equal to ωa. Common errors in this part were to use the ratio 1:2 or to fail to square the 2 when using the formula for v^2.

The required time from A to B can be found in different ways. It is important to consider whether the appropriate expression for each of the distances being found is $a\sin\omega t$ or $a\cos\omega t$.

Question 3

Many candidates were able to formulate two simultaneous equations for the speeds of A and B after the first collision, by means of conservation of momentum and Newton's law of restitution. There were a few sign errors in the equations, and candidates are reminded that a diagram with masses and velocities, with magnitude and direction clearly marked, is invaluable in avoiding such errors. Having found these speeds, the process then needs to be repeated for the second collision, between B and C. In this case, since the speed of B after the collision is zero, the simultaneous equations lead to a quadratic equation in e, the coefficient of restitution.

The total kinetic energy lost by the three spheres as a result of the two collisions can be found in three ways: by finding the loss in kinetic energy of each sphere individually, or by finding the difference between the total initial kinetic energy and the total final kinetic energy, or by finding the kinetic energy lost in each collision.

Question 4

As in all questions of this type, candidates are well advised to first identify all the forces acting on the rod, preferably showing them on the given diagram, or on their own diagram. This will help the candidate to ensure that they include all the relevant forces when taking moments or resolving. Candidates were required to find the tension in the string and the value of the coefficient of friction, in either order, leaving them free to choose their own method. The easiest and most direct method of solution is first to take moments about the point A, thereby eliminating the forces at A and obtaining an equation involving only the tension T and the weight W. This enables T to be found in terms of W. The next step is to resolve the forces horizontally and vertically, leading to expressions for the frictional force and the reaction force at A and hence the value of the coefficient of friction.

Of course, it is possible to take moments about several other points and many candidates did indeed do so. Almost invariably, however, the candidates who did this were unsuccessful in isolating the forces that they required from the resulting moments equations.

Question 5

Almost all candidates were able to make an attempt at finding the moment of inertia of the object and did so by finding the moment of inertia for each of the three component parts: the rod, the ring and the hollow sphere. The last two of these required the use of the parallel axes theorem. Many candidates set out their work clearly, with each of the contributions to the total moment of inertia clearly identified. Such detail is important when the final result is given in the question. Candidates who simply write down a sum of terms run considerable risk, since an error in one term which still leads to the given correct answer does cast doubt on the validity of the whole process.

Question 6

This question on the geometric distribution produced good answers by many candidates. In the final part, the equation to be solved is $1-q^{N-1}<0.95$. A common error was to have the power of q as N. In other cases, some incorrect manipulation of the inequality sign frequently led to $N > 8.39$ rather than $N < 8.39$ leading to an answer of 9 instead of 8.

Question 7

Many candidates were able to find the distribution function of X for $1 \leqslant x \leqslant 3$, but not all completed the description by including the value 0 for $x<1$ and the value 1 for $x>3$. The lower and upper quartiles were found accurately by many candidates, and then subtracted to give the interquartile range as equal to 1.

Question 8

Most candidates recognised this as an example of a paired-sample t-test. The method involves finding the difference between the two times for each of the runners and using these differences as the sample test values. The sample mean and an estimate for the population variance for these differences form the basis of a small sample t-test.

A minority of candidates did not recognise the information as a paired sample and used instead a two-sample t-test. This is not an appropriate approach for this type of data.

Question 9

This question tests the appropriateness of the given probability density function as a fit to the given data. The expected frequencies were given in the table, and candidates were asked to verify just one of them. Most candidates were able to do this, the only error being a tendency to round prematurely. As with any verification, all the necessary working must be shown to a suitable degree of accuracy, in this case at least 3 decimal places.

The final expected frequency in the table is less than 5 and this means that the last two columns must be combined, giving 12.65, before the chi-squared test value is calculated. A significant minority of candidates omitted to do this.

Question 10

The method for the first part of this question involves finding the values of S_{xy} and S_{yy} in terms of q and then equating S_{xy}/S_{yy} to the gradient $\frac{5}{4}$ of the given regression line of x on y. Many candidates approached the solution in this way and worked accurately to obtain the integer value 5 for q. Other candidates applied the correct method but made errors in the algebraic manipulation.

The final two parts of the question were attempted by many candidates, using either the correct value of q or the value that they had found in **part (i)**.

Question 11 (Mechanics)

This optional question was attempted by only a small minority of candidates, but these attempts were usually of a very high standard. To find the speeds of P and Q after their collision required three steps: conservation of energy for P from its starting point to its lowest point (giving the speed of P immediately before the collision), conservation of energy for Q from its starting point vertically below O (giving the velocity of Q immediately after the collision), and conservation of momentum at the collision (giving the speed of P immediately after the collision).

The second part of the question required the use of conservation of energy for P from immediately after the collision to the point where it loses contact with the inner surface of the sphere, and the use of Newton's second law of motion at this point of lost contact.

Question 11 (Statistics)

This was by far the more popular choice of optional question and the solutions were of a high standard.

The given confidence interval is used to find an unbiased estimate for the population variance of cherries of Type A. Many candidates made good progress, but some made an incorrect choice for the tabular t-value. Others used a z-value, which is not appropriate for a sample size of only 8.

The significance test was usually carried out successfully and accurately. The instruction to assume that the population variances for the two types of cherry are equal implies that a pooled common variance estimate is appropriate. Only a very few candidates worked with the assumption of unequal population variances.